Math 206B Lecture 22 Notes

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1 Littlewood-Richardson Coefficients

1.1 Multiplying symmetric functions

Recall
$$s_\lambda = \sum_{A \in \mathrm{SSYT}(\lambda)} x^A, \qquad x^A = x_1^{\#1\text{s in } A} x_2^{\#2\text{s in } A} \cdots$$
We can multiply many of the different bases of Λ:
$$e_\lambda e_\mu = e_{\lambda \cup \mu},$$
$$h_\lambda h_\mu = h_{\lambda \cup \mu},$$
$$p_\lambda p_\mu = p_{\lambda \cup \mu}.$$
And multiplying $m_\lambda m_\mu$ is straightforward. What about multiplying Schur functions?

Let $|\mu| + |\nu| = 1$. Then
$$s_\mu s_\nu = \sum_{|\lambda|=n} c_{\mu,\nu}^\lambda s_\lambda$$
What are the coefficients $c_{\mu,\nu}^\lambda$?

Proposition 1.1. $c_{\mu,\nu}^\lambda \in \mathbb{N}$.

Proof. Let S^ν, S^λ be irreducible representations. Then $s_\mu s_\nu$ corresponds to $\mathrm{ind}_{S_k \times S_{n-K}}^{S_n} S^\mu \otimes S^\nu$. So $c_{\mu,\nu}^\lambda$ is the inner product of S^λ with this induced character. This is the dimension of the irreducible representation S^λ in this representation. $\square$

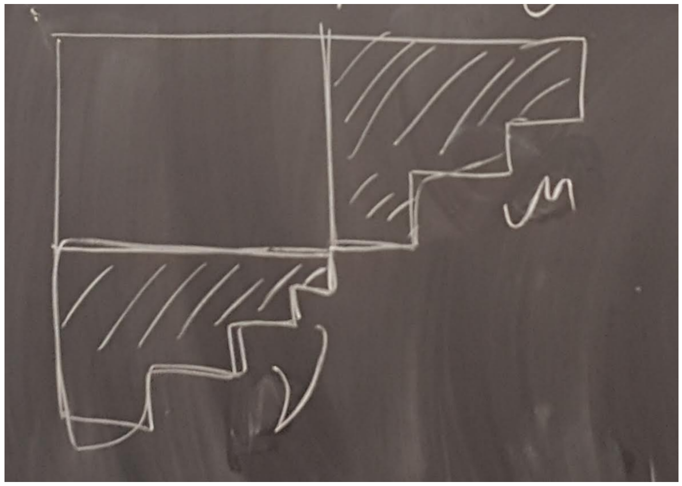
Theorem 1.1. *$c_{\mu,\nu}^\lambda = \# \, \mathrm{LR}(\lambda/\mu, \nu)$, the number of a certain type of semistandard Young tableaux.*

This is difficult to prove.[1]

[1] It is so difficult that Stanley did not actually prove it in his textbook.

1.2 Multiplying Schur functions

Let $\mu \circ \nu$ be the skew shape



Then

$$s_\mu s_\nu = s_{\mu\circ\nu} = \sum_{A\in\text{SSYT}} x^A = \sum_{|\lambda|=n} c^\lambda_{\mu,\nu} s_\lambda$$

How do we determine a tableau with shape $\mu \circ \nu$? Take the skew-shape and reduce it using Jeu-de-taquin.

Example 1.1. We reduce the skew tableau

		1	1
		3	
1	2		
3	3		

to the tableau

1	1	1
2	3	3
3		

So $c^\lambda_{\mu,\nu}$ is the multiplicity of any $P \in \text{SSYT}(\lambda)$ as a jeu-de-taquin of $B \circ C$, where $B \in \text{SSYT}(\mu)$ and $C \in \text{SSYT}(\nu)$.

Corollary 1.1. *$c^\lambda_{\mu,\nu} \in \#P$.*

There is a polynomial algorithm, jeu-de-taquin, for determining if B and C produce the correct tableau. But this is a very messy combinatorial interpretation. There is a better interpretation.

1.3 Ballot sequences

Definition 1.1. $(a_1, \ldots, a_n)$ is a **ballot sequence** if for all $k \in [n]$, the number of is among $a_1, \ldots, a_k$ is greater than the number of $(i+1)$s among $a_1, \ldots, a_k$ for all i.

Example 1.2. The sequence $(1,1,2,1,1,2,3,3,1,2,3)$ is a ballot sequence.

$\mathrm{Cat}(n)$ is the number of ballot sequences with n 1s and n 2s. Young tableau are basically the same as ballot sequences; if the number i in our tableau is in row j, we can make the i-th term in the sequence j.

When we have a pair of tablueux that we arrange into a skew shape, form a sequence by listing the numbers in each row from left to right, going down in rows.

Example 1.3.

$$\begin{array}{cccc} & & 1 & 1 \\ & & 3 & \\ 1 & 2 & & \\ 3 & 3 & & \end{array}$$

gives us the sequence $(1,1,3,3,1,3,3)$.

Theorem 1.2. *$c^{\lambda}_{\mu,\nu} = \#\,\mathrm{SSYT}(\nu, \lambda \setminus \mu)$ such that the sequence obtained from $B \circ C$ is a ballot sequence, where $B \in \mathrm{SSYT}(\mu)$ and $C \in \mathrm{SSYT}(\nu)$.*

Next time we will discuss the following.

Corollary 1.2. *$c^{\lambda}_{\mu,\nu}$ is the number of integer points in a polytope defined by the vectors λ, μ, ν.*

Theorem 1.3. *It can be determined in polynomial time whether $c^{\lambda}_{\mu,\nu} = 0$.*